The Wilsonian Flux

Issue 2 / January 2025

Welcome back to Issue 2 of the Wilsonian Flux, and a happy new year! I've had a lot of interest expressed in writing for the publication, and so you can expect issues to contain articles written by other students as well from now on - if you'd like to write, feel free to contact me. Enjoy the read!

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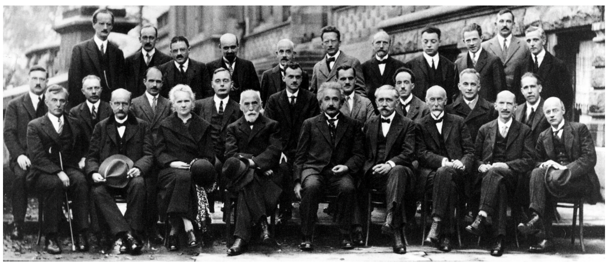


Figure 1: The 1927 Solvay Conference, titled "Electrons and Photons", where the world's most notable physicists met, including, but not limited to Schrödinger, Pauli, Heisenberg, Dirac, de Broglie, Bohr, Planck, Einstein and Curie [1]

1 An introduction to AC circuits

Written by **Vivaan, Year 12**

1.1 Introduction

Almost all circuits encountered up to A Level are direct current circuits. I'll now introduce alternating current circuits, where the current and voltage are varying sinusoidally. This is an extremely useful skill, given that mains power in the UK is AC. There are three main components that form a major part of AC circuit analysis: resistors, capacitors and inductors. Given that I've probably listed those in order of decreasing familiarity, I'll give a brief introduction to them soon.

1.2 Sinusoidal Voltage

In AC circuits, the voltage varies sinusoidally:

$$V(t) = V_{\text{max}} \sin(\omega t + \phi) = V_{\text{max}} e^{i(\omega t + \phi)}$$

Here, V_{max} is the peak voltage (which would be the amplitude of the sinusoid), ω is the angular frequency, and ϕ denotes a phase shift, meaning that voltage does not have to be 0 at $t = 0$, so the sinusoid can be shifted along the time axis. Note that we could have also used a cosine wave, and introduced a phase shift of $\frac{\pi}{2}$, to reach an equivalent sinusoid. As a consequence, we can also represent the voltage as a phasor in complex exponential form.

Another important value for any AC voltage is the rms, or root mean square, voltage. This simply refers to the "equivalent" DC voltage; in other words, given a certain time period, what DC voltage would lead to the same amount of energy being lost in the circuit as the AC voltage. We can calculate it as the average value of the square of the function over a period T and then taking the positive squareroot:

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T V_{\text{max}}^2 \sin^2(\omega t) dt}$$

$$V_{\text{rms}} = \sqrt{\frac{V_{\text{max}}^2}{T}\int_0^T \frac{1}{2}\left(1 - \cos(2\omega t)\right) dt}$$

$$V_{\text{rms}} = \sqrt{\frac{V_{\text{max}}^2}{2T}\left[t - \frac{1}{2}\sin(2\omega t)\right]_{t=0}^{t=T}}$$

$$V_{\text{rms}} = \sqrt{\frac{V_{\text{max}}^2}{2T}\left(T - \frac{1}{2}\sin(2\omega T)\right)}$$

$$V_{\text{rms}} = \sqrt{\frac{V_{\text{max}}^2}{2T}\left(T - \frac{1}{2}\sin(4\pi)\right)}$$

$$V_{\text{rms}} = \sqrt{\frac{V_{\text{max}}^2}{2}} = \frac{V_{\text{max}}}{\sqrt{2}}$$

Therefore, we have derived the relation that for an AC voltage, $V_{\text{rms}} = V_{\text{max}} \div \sqrt{2}$. The jump from $2\omega t$ to 4π might be confusing, but recall that $\omega = \frac{2\pi}{T}$. The same logic can be applied to calculate an rms current.

Phasors are used just as they are in other contexts with waves to describe sinusoidal voltage and current. As such we can consider the "lag" between two different sinusoids to

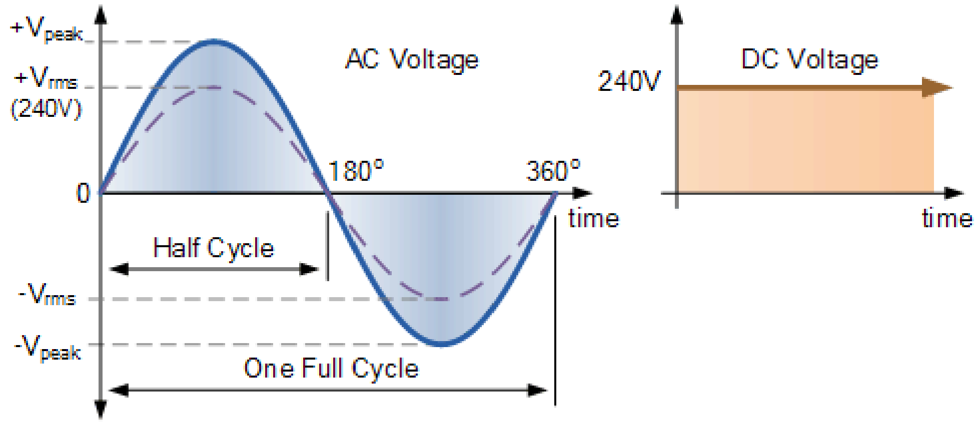


Figure 2: The rms voltage is the "equivalent" DC voltage [2]

be the angle between their phasors at any time. Just as a quick reminder, phasors rotate anticlockwise from the positive x-axis. By convention, the length of the voltage phasor represents the rms voltage. Furthermore, we can represent phasors as complex numbers, in either Cartesian form, or modulus-argument form - both representations lend themselves to different applications. We can also represent a complex number in exponential form, using Euler's formula: $e^{ix} = \cos x + i \sin x$.

1.3 Resistors, Capacitors and Inductors

We can now consider the behaviour of circuits containing passive components such as resistors, capacitors and inductors. It'll be necessary to discuss new concepts such as impedance and reactance, which didn't exist in DC circuits.

AC Resistance

As defined by Ohm's Law, we know that the voltage across an ohmic resistor is directly proportional to its resistance and current flowing through it. For AC circuits containing only resistors and an AC source, we can simply adapt Ohm's Law:

$$V(t) = I(t)R = I_{\text{max}} e^{i\omega t}$$

Now we are able to calculate the current flowing through a resistor at a given point in time, as well as the voltage across it. So for a purely resistive circuit the alternating current flowing through the resistor varies in proportion to the applied voltage across it following the same sinusoidal pattern. As the supply frequency is common to both the voltage and current, their phasors will also be common resulting in the current being "in-phase" with the voltage. Using our knowledge of the rms voltage and current, we can say that for a resistor, $R = V_{\text{rms}} \div I_{\text{rms}}$. This relationship between voltage and current was simply called resistance in DC circuits.

For AC circuits, as we generalise to other components, we call the relationship between peak voltage and peak current, the impedance. Impedance is given the symbol Z.

$$Z = \frac{V_{\text{peak}}}{I_{\text{peak}}}$$

The unit of Z is the ohm (Ω). Therefore, for a resistor in a DC circuit or an AC circuit, we can say that $R = Z$. This will not be true for other components such as capacitors and inductors, where the impedance will also have a imaginary part, called the reactance, X. The impedance is the vector sum of the two: $Z = \sqrt{R^2 + X^2}$. In other scenarios, it makes sense to write the impedance as ac complex number: $Z = R + iX$. As such we can represent the impedance of a component in an AC circuit in the following equivalent forms:

$$Z = \sqrt{R^2 + X^2} = R + iX = \left(\sqrt{R^2 + X^2}\right) e^{i\arctan\left(\frac{Y}{X}\right)} = Z_0 e^{i\phi}$$

Impedance follows the same laws as resistance in a circuit: that is, impedances in series are summed and the sum of the reciprocals of the impedances across parallel branches gives the reciprocal of the effective impedance of the parallel branches together.

AC Capacitance

Capacitors are electrical components which store energy on their plates in the form of electrical charge. The unit for capacitance is Farad (F). When a capacitor is connected across a DC supply voltage it charges up to the value of the applied voltage at a rate determined by its time constant and will maintain or hold this charge indefinitely as long as the supply voltage is present.

When connected to an AC source, the voltage across a capacitor "lags" $\frac{\pi}{2}$ behind the current. This is because the capacitor resists changes in voltage - it first has to charge or discharge to accommodate the varying voltage.

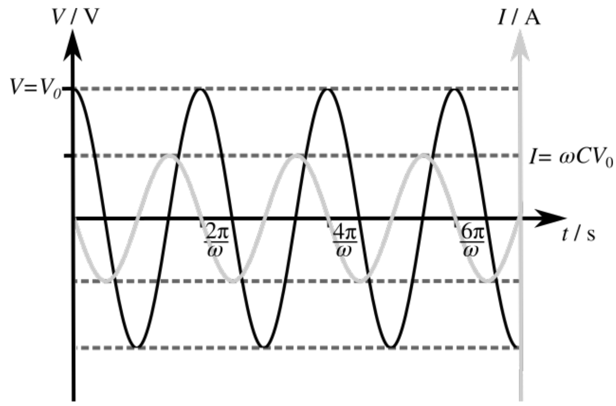


Figure 3: Voltage and current for a capacitor in an AC circuit, where the darker line is the voltage and lighter line is the current [3]

The opposition to current flow is called reactance. Let's derive the equation for the impedance of a capacitor:

$$\begin{aligned}
I(t) &= C\frac{dV}{dt} \\
I(t) &= C\frac{d}{dt}\left(e^{i\omega t}\right) \\
I(t) &= i\omega C\left(e^{i\omega t}\right) \\
I(t) &= i\omega C V(t) \\
Z &= \frac{V(t)}{I(t)} = \boxed{\frac{1}{i\omega C}} = \frac{1}{2i\pi f C}
\end{aligned}$$

Recall that the imaginary component of the impedance was called the reactance. Since there is no real part to the impedance of a capacitor, we can simply say that it has a purely reactive impedance. The reactance of the capacitor will decrease as the frequency of the supply voltage increases. Let's derive the fact that the voltage lags a quarter cycle behind the current. We know that the impedance can be given as $\frac{1}{i\omega C}$, so then:

$$\frac{1}{i\omega C} = \frac{1}{\omega C}\frac{1}{i} = \frac{1}{\omega C}\frac{1}{e^{i\frac{\pi}{2}}} = \frac{1}{\omega C}e^{-i\frac{\pi}{2}}$$

The $-\frac{\pi}{2}$ term indicates to us that the voltage is a phase angle of $\frac{\pi}{2}$ behind the current for a capacitor.

AC Inductance

Inductors are probably the least familiar component that we have discussed so far. Just as a capacitor stores energy in the form of electrical charge, an inductor stores its energy as a magnetic field. The simplest inductor is simply a coil of wire, often wrapped around a ferromagnetic material to increase a property called inductance. You may be familiar with inductors in the context of transformers, which is essentially two inductors connected to two different circuits. An inductor can also act as an electromagnet.

An inductor tends to resist changes in the current flowing through it. This is because the magnetic flux through the coil due to the current decreases as the current decreases, and due to Faraday's and Lenz's Laws, there will be an electromotive force (EMF) acting to oppose this change. The inductance of an inductor is measured in Henries (H). One henry is defined as the amount of inductance which, when the rate of change of current is one ampere per second, will induce an emf of one volt. This leads us to the definition of inductance:

$$V = L\frac{dI}{dt} \longrightarrow L = \frac{V}{\frac{dI}{dt}}$$

The behaviour of an inductor in a DC circuit is in many ways analagous to that of a capacitor - I won't discuss this in too much detail as I'll focus on how they behave in AC circuits.

Let's see if we can derive an expression for the impedance of an inductor, similar to how we did for a capacitor:

$$\begin{aligned}
V(t) &= L\frac{dI}{dt} \\
\int V_0 e^{i\omega t} dt &= L\int \frac{dI}{dt}dt \\
\frac{V_0}{i\omega}e^{i\omega t} &= LI(t) \\
\frac{1}{i\omega}V(t) &= LI(t) \\
Z = \frac{V(t)}{I(t)} &= \boxed{i\omega L} = 2i\pi f L
\end{aligned}$$

Much like a capacitor, the impedance of an inductor is purely reactive. However, unlike a capacitor, we can now see that the voltage for an inductor leads the current by a quarter cycle instead. Knowing the impedance to be $i\omega L$, we see:

$$i\omega L = L\omega e^{i\frac{\pi}{2}}$$

We see that the $\frac{\pi}{2}$ term in the exponent indicates that the voltage is a phase angle of $\frac{\pi}{2}$ ahead of the current.

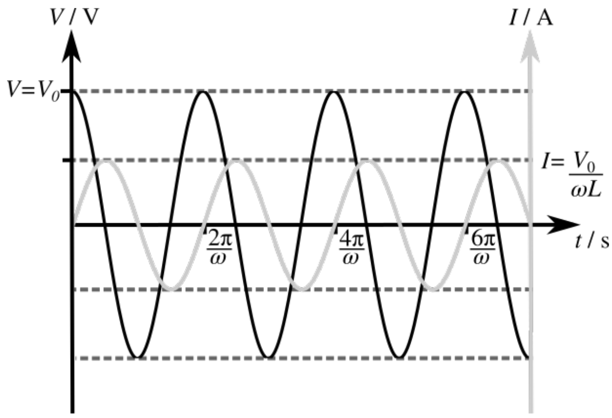
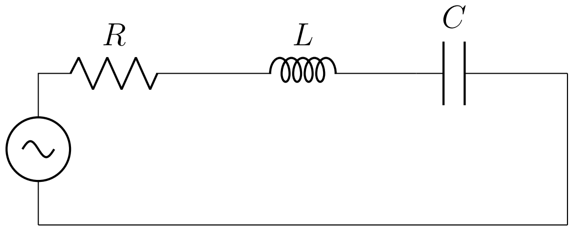


Figure 4: Voltage and current for a inductor in an AC circuit, where the darker line is the voltage and lighter line is the current [3]

1.4 RLC Circuit Analysis

Armed with knowledge about the impedances of these 3 different components, we can tackle AC circuits containing them. This would best be demonstrated through an example. Consider the following circuit:



Here, the supply voltage supplies a maximum of $V_0 = 100$V and operates at $f = 50$Hz. The resistance $R = 12\Omega$, inductance $L = 0.15$H and capacitance $C = 100\mu$F. Calculate the impedance of the circuit and the maximum current within the circuit.

We first calculate the resistances and reactances of each component:

$$\begin{aligned} R &= R = 12\Omega \\ X_L &= 2\pi f L = 47.13\Omega \\ X_C &= \frac{1}{2\pi f C} = 31.83\Omega \end{aligned}$$

We treat the capacitive reactance as negative, since it causes the voltage to lag behind the current, and since the impedance is the vector sum of the resistances and reactances, we can calculate the impedance.

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \boxed{19.4\Omega}$$

We can now calculate the maximum current.

$$I_{\text{max}} = \frac{V_{\text{max}}}{Z} = \frac{100\text{V}}{19.4\Omega} = \boxed{0.619\text{A}}$$

2 Fourier Transforms

Written by **Vivaan, Year 12**

2.1 Transforms, signals and systems

Transforms are essential in a lot of mathematical and physical contexts, among a variety of problems. They're simply a mathematical operation that changes the representation of a function or a signal, usually through a change in domain - think of it like a switch in perspective, for example, switching from the time domain (how things change over time) to the frequency domain (how things behave across different frequencies).

It would also make sense to briefly define what a signal is. In essence, signals refer to a function that represents how something changes over time, such as voltage, sound or temperature. A system is something that modifies a given signal, such as an amplifier or filter. The ability to interpret and analyse signals is a central challenge in a lot of contexts, as well as accurately predicting the effect that a system would have on a signal. This is where transforms are particularly useful: they can reveal patterns and properties of the original signal that may not be obvious in their initial representation.

I'll introduce an important transform you're almost guaranteed to come across in the future in any maths, physics or engineering discipline: the Fourier transform. In the future, we might cover another significant transform called the Laplace transform - which is closely related to the Fourier transform.

2.2 Fourier series and the Fourier Transform

The Fourier Transform is conceptually quite simple, even if the math can look daunting at first. Given a certain signal, it can represent it as a sum of sines and cosines; for example, given a sound wave, a Fourier transform of the sound wave would give you its component frequencies, their amplitudes, and the offsets of each - it's like finding the ingredients for a signal. The applications of this are pretty exciting: you could boost the bass of a given sound wave, or filter through a radio wave in order to only listen to a given frequency.

First, we will discuss what a Fourier series is, before generalising to the Fourier transform. The Fourier series for any periodic function (that is $f(x+a) = f(x)$) is simply its representation as a (sometimes infinite) sum of sines and cosines:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{2\pi nx}{a}\right) + \sum_{m=0}^{\infty} \beta_m \sin\left(\frac{2\pi mx}{a}\right)$$

The coefficients α_n and β_m are referred to as the Fourier coefficients and can be calculated for any given x. The question of whether we can do this for any arbitrary periodic function is rather intricate, but as a general rule of thumb, if $\int_{-\infty}^{\infty} |f(x)|^2 dx$ exists and is finite, then the Fourier series will converge. Functions for which this property holds are referred to as "square-integrable". Using Euler's formula, we can manipulate the above equation into the following summation:

$$f(x) = \sum_{-\infty}^{\infty} f_n e^{\frac{2\pi inx}{a}}$$

Instead of a sum over sines and cosines we have a sum of complex exponentials, which is much neater. Also notice that we now have a new set of coefficients for each exponential, and therefore if the Fourier coefficients are known, we can calculate $f(x)$ using the above summation.

Conversely, given a function $f(x)$, we should be able to calculate its Fourier series, which in turn means calculating the Fourier coefficients for each n. We must quickly define the Kronecker Delta, as follows:

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Now observe that:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-\frac{2\pi imx}{a}} e^{\frac{2\pi inx}{a}} dx = a\delta_{mn}$$

It is a beneficial exercise to convince yourself of this result for $m, n \in \mathbb{Z}$ - consider how the exponents add and the fact that the function is periodic. Therefore, we can now use this fact for any general function $f(x)$, for which we are trying to find the corresponding Fourier series.

$$\begin{aligned}
\int_{-\frac{a}{2}}^{\frac{a}{2}} \left(e^{-\frac{2\pi imx}{a}}\right) f(x)dx &= \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} dx\right) \left(e^{-\frac{2\pi imx}{a}}\right) \left[\sum_{n=-\infty}^{\infty} e^{\frac{2\pi inx}{a}} f_n\right] \\
&= \sum_{n=-\infty}^{\infty} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(e^{-\frac{2\pi imx}{a}} e^{\frac{2\pi inx}{a}} dx\right) f_n \\
&= \sum_{n=-\infty}^{\infty} a\delta_{mn} f_n \\
&= af_m
\end{aligned}$$

By multiplying the function by $e^{-\frac{2\pi imx}{a}}$, we essentially filter out all other components of the function, keeping only the Fourier component with the matching index m, and finding out its Fourier coefficient f_m.

Thus we arrive at a set of relations to express a signal $f(x)$ in term of its Fourier components and to go the other way as well.

$$\begin{aligned}
f(x) &= \sum_{-\infty}^{\infty} e^{ik_n x} f_n \\
f_n &= \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{ik_n x} f(x)dx \\
\text{where } k_n &= \frac{2\pi n}{a}
\end{aligned}$$

The numbers k_n are called the "wave-numbers" and they are quantised to integer multiples of $\frac{2\pi}{a}$.

2.3 A square wave

Enough theory - let's see if we can represent a square wave as a Fourier series. Let's define our square wave function's behaviour by making it negative before $x = 0$ and jumping to positive on the y-axis. We will give the function an arbitrary period a.

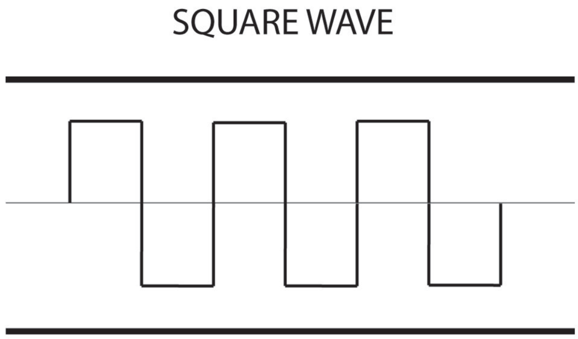


Figure 5: A square wave [4]

$$f(x) = \begin{cases} -1 & -\frac{a}{2} \leq x < 0 \\ +1 & 0 \leq x < \frac{a}{2} \end{cases}$$

Let us now compute the Fourier coefficients:

$$\begin{aligned}
f_n &= \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{ik_n x} f(x) dx \\
&= \frac{1}{a} \left(\int_{-\frac{a}{2}}^{0} e^{ik_n x}(-1) dx + \int_{0}^{\frac{a}{2}} e^{ik_n x}(1) dx \right) \\
&= \frac{1}{a} \left(-\int_{-\frac{a}{2}}^{0} e^{ik_n x} dx + \int_{0}^{\frac{a}{2}} e^{ik_n x} dx \right) \\
&= \frac{1}{a} \left(-\left[\frac{e^{ik_n x}}{ik_n} \right]_{-\frac{a}{2}}^{0} + \left[\frac{e^{ik_n x}}{ik_n} \right]_{0}^{\frac{a}{2}} \right) \\
&= \frac{1}{a} \left(-\frac{1}{ik_n} \left(1 - e^{-\frac{ik_n a}{2}} \right) + \frac{1}{ik_n} \left(e^{\frac{ik_n a}{2}} - 1 \right) \right) \\
&= \frac{1}{aik_n} \left(2 - \left(e^{-i\pi n} + e^{i\pi n} \right) \right) \\
&= \frac{1}{aik_n} \left(2 - 2\cos(\pi n) \right) \\
&= \frac{2}{aik_n} \left(1 - \cos(\pi n) \right) \\
&= \frac{4}{aik_n} \sin^2 \left(\frac{\pi n}{2} \right) \\
&= \frac{4}{\pi n i} \sin^2 \left(\frac{\pi n}{2} \right)
\end{aligned}$$

Using this result, we can differentiate between odd and even values for n.

$$f_n = \begin{cases} \frac{2}{\pi n i} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

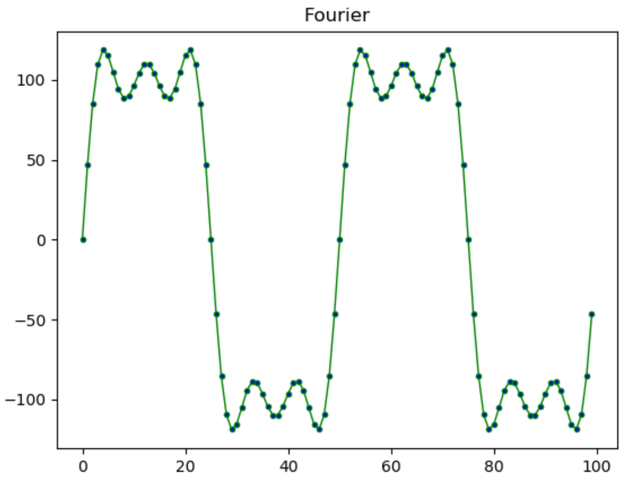


Figure 6: The first three terms of the Fourier series for a square wave [5]

Finally, we can represent the square wave as an infinite Fourier series. We notice that because the function is odd and symmetric about $x = 0$, all the cosine terms in its expansion will cancel out, leaving only sine terms. Furthermore, it is important to note that f_n and f_{-n} are complex conjugates, which results in a purely real sine series, with coefficients doubled from what you would expect them to be.

$$\begin{aligned} f(x) &= \frac{4}{\pi}\sin\left(\frac{2\pi x}{a}\right) + \frac{4}{3\pi}\sin\left(\frac{6\pi x}{a}\right) + \frac{4}{5\pi}\sin\left(\frac{10\pi x}{a}\right) + \dots \\ &= \boxed{\frac{4}{\pi}\sum_{n=1,3,5\dots}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi}{a}\right)} \end{aligned}$$

We have now successfully represented a square wave as an infinite Fourier summation. Adding more terms to the summation would lead it to be closer and closer to the actual square wave, but otherwise this is technically only an approximation, similar to Maclaurin and Taylor series.

3 Partial Derivatives

Written by **Adhrit, Year 12**

3.1 The motivation behind partial derivatives

We all know what derivatives are. The standard $\frac{dy}{dx}$. The limit as h approaches 0 of $\frac{f(x+h)-f(x)}{h}$. It gives the gradient of a function. More generally, the rate at which one variable is changing with respect to another. It can be used to model how quickly a population is growing per given unit of time, how much a tank is emptying by per unit time, maximising the volume of an object from a given area of material. The options are endless. But what happens when you have a function that's dependent on many variables? The normal differential operator doesn't work, and it is here where partial derivatives come in.

The standard differential operator $\frac{d}{dx}$, or otherwise expressed as $f'(x)$ tells us what we are differentiating with respect to. But what happens when we have a function that's dependent on both x and y? If we have a function $f(x)$ and we plot $y = f(x)$ against the input x, a standard curve on a 2D plane is obtained. However if we have a function $f(x, y)$ plotted against a 3rd variable z, what we obtain is a surface in space. Now as there are two variables, the derivative with just respect to x or y can't be taken. So how do we get the gradient, or rate of change, of this surface? The answer to this is partial derivatives. The partial derivative of $z = f(x, y)$ with respect to x can be written in the following equivalent forms:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x, y)) = z_x = \frac{\partial z}{\partial x} = D_x f$$

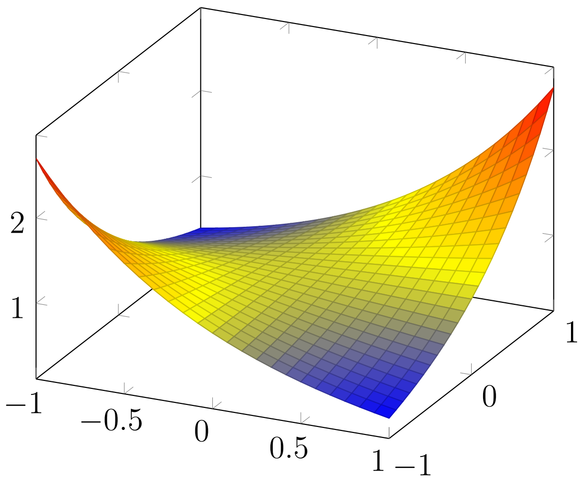
3.2 Computing partial derivatives

If you are comfortable with differentiation with a single variable, partial derivatives won't be a significant jump. When differentiating a multivariable function with respect a variable, you simply have to treat all the other variables as a constant. For example, given a function $f(x, y) = e^{xy} + \sin(x^2 y)$, let's compute the partial derivatives with respect to both x and y. So to get $\frac{\partial f}{\partial x}$, take the normal derivative with respect to x treating y as a constant:

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= \frac{\partial}{\partial x}(e^{xy} + \sin(x^2 y)) \\ &= \frac{\partial}{\partial x}(e^{xy}) + \frac{\partial}{\partial x}(\sin(x^2 y)) \\ \frac{\partial f}{\partial x} &= \boxed{ye^{xy} + 2xy\cos(x^2 y)} \end{aligned}$$

We can calculate the partial derivative with respect to y, by treating x as a constant:

$$\begin{aligned} \frac{\partial}{\partial y} f(x, y) &= \frac{\partial}{\partial y}(e^{xy} + \sin(x^2 y)) \\ \frac{\partial f}{\partial y} &= \boxed{xe^{xy} + x^2\cos(x^2 y)} \end{aligned}$$



Here is the graph of $f(x, y)$. We can also compute higher order derivatives, just as we could with single variable functions. However, the extra complexity lies in the fact that we can compute the derivative with respect to one variable first, and then with a different variable the second time. Let's demonstrate this and see if any interesting results arise.

3.3 Clairaut's Theorem

Given a function of two variables $f(x, y)$ it is possible to compute a total of 4 different partial derivatives, depending on the order that we differentiate in:

- Differentiate with respect to x twice: $\frac{\partial^2 f}{\partial x^2}$
- Differentiate with respect to y twice: $\frac{\partial^2 f}{\partial y^2}$
- Differentiate with respect to x then with respect to y: $\frac{\partial^2 f}{\partial y \partial x}$
- Differentiate with respect to y then with respect to x: $\frac{\partial^2 f}{\partial x \partial y}$

Note the order matters when writing mixed derivatives, and that in different notation the order may be different, that is: $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

Now let's carry on with our example from earlier, and compute both of the mixed derivatives, by differentiating $\frac{\partial f}{\partial x}$ with respect to y to obtain $\frac{\partial^2 f}{\partial y \partial x}$ and by differentiating $\frac{\partial f}{\partial y}$ with respect to x to obtain $\frac{\partial^2 f}{\partial x \partial y}$.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial x}(ye^{xy} + 2xy\cos(x^2 y)) \\
&= \frac{\partial}{\partial y}(ye^{xy}) + \frac{\partial}{\partial y}(2xy\cos(x^2 y)) \\
&= \boxed{e^{xy} + xye^{xy} + 2x\cos(x^2 y) - 2x^3 y\sin(x^2 y)}
\end{aligned}$$

Note that we used the product rule for differentiation in the last line, and it works analogously to how it would with a single variable. Now let's compute the other mixed derivative:

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial x}(xe^{xy} + x^2\cos(x^2 y)) \\
&= \frac{\partial}{\partial x}(xe^{xy}) + \frac{\partial}{\partial x}(x^2\cos(x^2 y)) \\
&= \boxed{e^{xy} + xye^{xy} + 2x\cos(x^2 y) - 2x^3 y\sin(x^2 y)}
\end{aligned}
$$

And we have thus demonstrated Clairaut's Theorem, that is: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. This can easily be generalised to functions of higher dimensions e.g. $f_{xyzt} = f_{txzy}$, or even with higher order derivatives that involve differentiating with respect to the same variable multiple times e.g. $f_{xyzxxyz} = f_{xxxyyzz}$. This is subject to some continuity conditions, but this will work for all "nice" functions. Using this can sometimes making finding higher-order derivatives of multivariable functions easier than it would be by differentiating in a different order. This result is also known as Schwarz's Theorem and Young's Theorem, as well as Clairaut's Theorem. For further reading, look into the Hessian matrix and convince yourself using Clairaut's Theorem that the Hessian matrix for any multivariable function is always symmetric along its diagonal.

3.4 Directional derivatives

Let's consider how we would go about finding the gradient of a 3D function. It is important to note that the meaning of "gradient" tends to be slightly less specific than the 2D version, since there are now multiple different directions that we could travel in on the plane, each with a different rate of change. Finding just one partial derivative assumes that you are holding the other variables constant, but what if we don't want to do this.

As a result, the gradient is simply a vector made up of all the partial derivatives of a function. For a function $f(x, y)$ the gradient, typically denoted as $\vec{\nabla} f$ is equal to $f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}}$. So the partial derivatives become the corresponding components of a vector in it's direction. To find the gradient of a function, we need to define what direction we are going in, and as a result a directional derivative can only be found with a unit vector in mind - this unit vector will be the direction that we are travelling in.

This can be extended to more dimensions, if you have a function $f(x, y, z, ...)$, $\vec{\nabla} f$ will be $f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}} + f_z\hat{\boldsymbol{k}} + ...$ where the resulting vector ends up being the the slope of the function at any given point.

For example, given $f(x, y, z) = e^x z^6 \cos(y)$, then:

$$
\vec{\nabla} f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} e^x z^6 \cos(y) \\ -e^x z^6 \sin(y) \\ 6e^x z^5 \cos(y) \end{pmatrix}
$$

Now to find the directional derivative in the direction of a unit vector $\vec{u} = a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}$, we can simply compute $\vec{\nabla} f \cdot \vec{u}$. This is represented $D_{\vec{u}} f$ Say we wanted to find the gradient at a point (x_0, y_0, z_0) in the direction of $\vec{u}$, we would simply evaluate $(f_x(x_0)\hat{\boldsymbol{\imath}} + f_y(y_0)\hat{\boldsymbol{\jmath}} + f_z(z_0)\hat{\boldsymbol{k}}) \cdot \vec{u}$. Note that we can now consider first order partial derivatives to just be directional derivatives in the direction of the basis vectors; that is, f_x is simply the directional derivative in the direction of $\hat{\boldsymbol{\imath}}$. This works similarly for f_y and f_z.

The maximum value of the directional derivative $D_{\vec{u}} f$ at $\vec{r}$ is $|\vec{\nabla} f(\vec{r})|$ and in the direction given by $\vec{\nabla} f(\vec{r})$. This allows us to do a lot of things such as finding the maximum rate of change of function at a given point, and the direction we need to move in to achieve that maximum rate of change.

3.5 The wave equation

So what examples of partial derivatives exist in our applied physical world then? Perhaps one of the most famous instances where they are involved is in the wave equation, which a second order partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This tells us for some wave function $u(x, t)$ the second partial derivative of the wave with respect to time is equal to the speed of the wave squared multiplied by the second partial derivative of the wave with respect to space. For a three dimensional representation, it can be extended to the three dimensions of space (and time):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 (\nabla^2 u)$$

It is imperative to note that this model was developed with general assumptions, so different assumptions will lead to different variations and models. This is just a general equation. Here we also introduce the Laplacian, ∇^2, which is the sum of the second order partial derivatives with respect to each variable - it has many useful properties.

The shape within which the waves arise in higher dimensions or domains (like 3) can be complicated, but for any chosen shape of domain, they are functions analogous to Bernoulli's sines and cosines - the simplest patterns of vibration. These patterns are called modes, and all other waves can be obtained by superposing these normal modes, with using infinite series if necessary. These principles can be applied to many sorts of waves: water, sound, electromagnetic, even quantum waves, but also very interestingly, earthquakes.

Sophisticated versions of the wave equation which were created using different modelling assumptions allowed scientists to analyse pressure and shear waves (P and S waves) that were generated from earthquakes. From this, an understanding of what was going on thousands of kilometers below the Earth's surface. Aside from the analysis leading to the realisation of Earth having a liquid outer core and a solid inner core, analysis derived from the wave equation led to the mapping of tectonic plates as they slide beneath each other (subduction) which is what causes the earthquakes. Recently, it was found that plates don't need to subduct as a whole, but can break into gigantic slabs and sink back into the mantle at different depths, causing earthquakes. The biggest aim in this field is to use these models and future ones to be able to predict when these tremors will occur, allowing the area to be appropriately prepared and evacuated potentially saving hundreds and thousands of lives. Currently, it still all seems pretty elusive, partially due to the fact that the conditions that trigger earthquakes are a combination of several factors spread over several locations, making it hard to combine these to make predictions. However, progress is being made, and seismologists using the wave equation are underpinning several other methods in trying to be able to make these life-saving future predictions.

4 Problems

AC Circuits

1. Explain why at high frequencies a capacitor acts as an ac short, whereas an inductor acts as an open circuit.
2. In an RLC series circuit, can the voltage measured across the capacitor be greater than the voltage of the source? Answer the same question for the voltage across the inductor.
3. An RLC series circuit with $R = 600\Omega$, $L =$ 30mH and $C =$ 0.050μF is driven by an AC source whose frequency and voltage amplitude are 500Hz and 50V, respectively.
   (a) What is the impedance of the circuit?
   (b) What is the amplitude of the current in the circuit?
   (c) What is the phase angle between the emf of the source and the current?

Fourier Transform

1. Sketch the graph of the integral of the square wave function.
2. Find an expression for this function and consequently find the Fourier series for it - there's a short way and a long way.
3. Evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Partial Derivatives

1. Find $\frac{\partial z}{\partial x}$ for:

$$x^2 \sin(y^3) + xe^{3z} - \cos(z^2) = 3y - 6z + 8$$

2. Given $f(x, y, z) = x^4y^3z^6$, find $\frac{\partial^6 f}{\partial y \partial z^2 \partial y \partial x^2}$
3. Find the maximum rate of change of the function $f(x, y) = \sqrt{x^2 + y^4}$ at $(-2, 3)$.
4. Given $PV = nRT$, evaluate:

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}$$

5 Solutions

These are my solutions to the problems listed in Issue 1 (December 2024). Sometimes solutions can be just as interesting as problems, so if a particular solution catches your eye, I'd encourage you to go and see how you would have gone about solving the problem initially!

Exploring rotational motion

1. **Q: Explain why it is easier to balance a cup on a finger upside down, with your finger inside the cup, as opposed to a cup the right way up, with your finger under the base.**

   The difference in ease of balancing the cup is due to the position of the centre of mass of the cup relative to the pivot (where your finger meets the cup). With the whole cup above your finger, a small deviation causing the centre of mass to move off the line of action will cause a moment around your finger, causing the cup to topple even further, making it unstable. With your finger inside the cup, the centre of mass of the cup essentially lies within your finger. Any small movement causing the centre of mass will cause a moment about your finger, which will restore the cup to its equilibrium position. This restoring moment makes it easier to balance a cup this way round.

2. **Q: For two point masses with mass m_1 and m_2 separated by a distance l rotating around their common centre of mass, find their (a) moment of inertia and (b) angular momentum, if one full rotation takes a time period T**

   The distance of mass m_1 from the COM is

   $$r_1 = \frac{m_2}{m_1 + m_2} l$$

   and the distance of mass m_2 from the COM is

   $$r_2 = \frac{m_1}{m_1 + m_2} l$$

   (a) We can now find the moment of inertia:

   $$\begin{aligned} I &= \sum m_i r_i^2 \\ &= m_1 \left(\frac{m_2}{m_1 + m_2} l \right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} l \right)^2 \\ &= \frac{l^2 (m_1 m_2^2 + m_1^2 m_2)}{(m_1 + m_2)^2} \\ I &= \boxed{\frac{m_1 m_2}{m_1 + m_2} l^2} \end{aligned}$$

   (b) The angular momentum can also be quite concisely found:

   $$\begin{aligned} L &= I\omega \\ \omega &= \frac{2\pi}{T} \\ L &= \frac{2\pi I}{T} \\ L &= \boxed{2\pi \left(\frac{m_1 m_2}{m_1 + m_2} \right) \frac{l^2}{T}} \end{aligned}$$

3. **Q: Estimate the angular momentum of the Earth: (a) in its orbit around the Sun and (b) on its axis**

   (a) We can treat the Earth as a point mass at a certain distance from the sun. The difficulty in this problem lies in estimating values such as the mass of the Earth and the distance of the Earth from the sun. I'd encourage you to use any facts and formulae you know - how long does light take to travel to the Earth from the Sun, can you use this information to estimate the distance? How about using values and formulae you know about gravity to estimate the mass of the Earth. Calculating the angular velocity of the Earth around the sun is not a particularly difficult task - just 2π divided by a year. For completeness, here is the calculation outlined above using the true values:

   $$\begin{aligned} L &= (5.97 \times 10^{24} \text{ kg}) \times (1.496 \times 10^{11} \text{ m})^2 \times (1.99 \times 10^{-7} \text{ rad s}^{-1}) \\ &= \boxed{2.66 \times 10^{40} \text{ kg m}^2 \text{ s}^{-1}} \end{aligned}$$

   (b) Using a similar logic to the above, with the additional fact that the moment of inertia of a sphere around $I = \frac{2}{5}MR^2$, I'll write out the calculation below, where we use the Earth's mass, radius and its angular velocity as it spins on its axis.

   $$\begin{aligned} L &= \frac{2}{5} \times (5.97 \times 10^{24} \text{ kg}) \times (6.371 \times 10^{6} \text{ m})^2 \times (7.27 \times 10^{-5} \text{ rad s}^{-1}) \\ &= \boxed{7.05 \times 10^{33} \text{ kg m}^2 \text{ s}^{-1}} \end{aligned}$$

4. **Q: Prove that an unhinged, free body rotates about its centre of mass after a force has been applied to it.**

   The simplest way to "prove" this is essentially by contradiction, showing that this is the only possible way for the rigid body to move. In the absence of external forces, the center of mass of any collection of particles moves at a constant velocity. This is true whether they are stuck together in a single body or are just a bunch of separate bodies with or without interactions between them. We now move to a frame of reference moving at that velocity. In that frame the CoM is stationary.

   Now suppose that the particles are indeed stuck together to form a rigid body. We see that the body is moving so that: 1) the CoM remains fixed, 2) all the distances between the particles are fixed i.e. it is rigid.

   A motion with these two properties, (1) and (2), is precisely what is meant by the phrase "a rotation about the CoM". It would be impossible to have an axis of rotation which doesn't pass through the CoM, simply because this would result in the CoM exhibiting circular motion, contradicting our statement that in the absence of external forces, the centre of mass moves at a constant velocity.

5. **Q: Why might the cross-product definition for angular momentum break down in higher dimensions?**

   The full answer to this is quite mathematically rigorous, but essentially it relies on the fact that we have defined the right hand rule for vectors rather arbitrarily. In higher dimensions, there may be more than 2 possible vectors that are "perpendicular" or orthogonal to the original 2 vectors for which we are finding the cross product and so the right hand rule wouldn't work. The proper generalisation to an n-dimensional space would be to call angular momentum a bivector. The exact nature of bivectors is rather terse so I won't go into it, but feel free to look into it. As a fun fact, the 7-dimensional cross product is well-defined, unlike for 4, 5 and 6 dimensions. Have a look into why this is - it's related to octonions.

Coordinate systems

1. **Q: By integrating in a cylindrical co-ordinate system, find the volume of a frustum of a cone with base radius R, top radius r and height h.**

   The trick with this was to write the radius as a function of height, knowing that it varies linearly with height between r and R. Then set up the integral as normal.

$$
\begin{aligned}
r(z) &= R - \frac{R-r}{h}z \\
V &= \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=h}\int_{0}^{r(z)} r\,dr\,dz\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=h}\left[\frac{r^2}{2}\right]_0^{r(z)} dz\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=h}\frac{r(z)^2}{2}\,dz\,d\theta \\
&= \pi\int_{z=0}^{z=h} r(z)^2\,dz \\
&= \pi\int_{z=0}^{z=h}\left(R-\frac{R-r}{h}z\right)^2 dz \\
&= \pi\int_{z=0}^{z=h}\left(R^2-2R\frac{R-r}{h}z+\left(\frac{R-r}{h}\right)^2 z^2\right) dz \\
&= \pi\left[R^2 z - R\frac{R-r}{h}z^2+\frac{1}{3}\left(\frac{R-r}{h}\right)^2 z^3\right]_{z=0}^{z=h} \\
&= \pi\left(R^2 h - R(R-r)h + \frac{1}{3}(R-r)^2 h\right) \\
&= \pi h\left(R^2 - R^2 + Rr + \frac{1}{3}R^2 - \frac{2}{3}Rr + \frac{1}{3}r^2\right) \\
V &= \boxed{\frac{1}{3}\pi h(R^2+Rr+r^2)}
\end{aligned}
$$

2. **Q: Find the distance travelled by a particle in a time period τ , where the particle is travelling in a helical path described by the equations:** $r = R$, $\theta = \omega t$, $z = vt$

   For this we can simply use the arc length formula:

$$
s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt
$$

   First we must convert our functions into Cartesian coordinates:

$$
\begin{aligned}
x &= R\cos(\omega t) \\
y &= R\sin(\omega t) \\
z &= vt
\end{aligned}
$$

   And then simply compute from 0 until time τ:

$$
\begin{aligned}
s &= \int_0^\tau \sqrt{(-R\omega\sin(\omega t))^2+(R\omega\cos(\omega t))^2+v^2}\,dt \\
&= \int_0^\tau \sqrt{R^2\omega^2+v^2}\,dt \\
s &= \boxed{\tau\sqrt{R^2\omega^2+v^2}}
\end{aligned}
$$

3. **Q: Calculate the moment of inertia for a cone with mass M, base radius R and height L around the axis perpendicular to base, passing through the apex**

   The cone has uniform density so we can calculate this by diving its mass by its volume:

$$\rho = \frac{M}{\frac{1}{3}\pi R^2 L} = \frac{3M}{\pi R^2 L}$$

   In cylindrical coordinates, the volume element $dV = r\,dr\,d\theta\,dz$, so therefore the mass element $dm = \rho\,dV = \rho\,r\,dr\,d\theta\,dz$. We can now setup the integral:

$$\begin{aligned}
I &= \int r^2\,dm \\
&= \rho \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=L} \int_{r=0}^{r=\frac{R}{L}z} r^3\,dr\,d\theta\,dz \\
&= 2\pi\rho \int_{z=0}^{z=L} \int_{r=0}^{r=\frac{R}{L}z} r^3\,dr\,dz \\
&= 2\pi\rho \int_{z=0}^{z=L} \left[\frac{r^4}{4}\right]_{r=0}^{r=\frac{R}{L}z} dz \\
&= 2\pi\rho \int_{z=0}^{z=L} \frac{R^4}{4L^4} z^4\,dz \\
&= 2\pi \left[\frac{R^4}{20L^4} z^5\right]_{z=0}^{z=L} \\
&= \rho\frac{\pi R^4 L}{10} \\
&= \frac{3M}{\pi R^2 L} \cdot \frac{\pi R^4 L}{10} \\
I &= \boxed{\frac{3}{10} M R^2}
\end{aligned}$$

   Notice how the answer doesn't depend on the height of the cone.

4. **Find the value of I where $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$**

   This is actually a pretty famous problem, in fact, this integral is called the Gaussian integral. One way of solving it is by squaring the integral and then using polar coordinates:

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\right)
\end{aligned}$$

   Now, we can switch to polar coordinates, because $r^2 = x^2 + y^2$, but note that the area element $dx\,dy$ becomes $r\,dr\,d\theta$ in polar coordinates. We also need to change our bounds accordingly

$$\begin{aligned}
I^2 &= \left(\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr\,d\theta\right) \\
&= 2\pi \int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr
\end{aligned}$$

Here we employ a u-substitution where $u = r^2$ $(du = 2r\,dr)$and continue:

$$\begin{aligned} I^2 &= \pi \int_{r=0}^{r=\infty} e^{-u}\,du \\ &= \pi \left[-e^{-u}\right]_{u=0}^{u=\infty} \\ I^2 &= \pi \end{aligned}$$

And so we can arrive at a rather interesting result

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \boxed{\sqrt{\pi}}$$

5. **Which points in $\mathbb{R}^3$ have the same co-ordinates in all 3 co-ordinate systems: Cartesian, cylindrical and spherical?**

   It's possible to come to this conclusion mathematically, but the only point that has the same coordinate values in all 3 systems is the origin - $(0, 0, 0)$.

Solutions to the problems from this issue will feature in the next issue!

Credits

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